

Commutative Local Rings Whose Ideals Are Direct Sums of Cyclic Modules^{*†‡}

M. Behboodi^{a,b§} and S. H. Shojaee^a

^aDepartment of Mathematical Sciences, Isfahan University of Technology
P.O.Box: 84156-83111, Isfahan, Iran

^bSchool of Mathematics, Institute for Research in Fundamental Sciences (IPM)
P.O.Box: 19395-5746, Tehran, Iran
mbehbood@cc.iut.ac.ir
hshojaee@math.iut.ac.ir

Abstract

A well-known result of Köthe and Cohen-Kaplansky states that a commutative ring R has the property that every R -module is a direct sum of cyclic modules if and only if R is an Artinian principal ideal ring. This motivated us to study commutative rings for which every ideal is a direct sum of cyclic modules. Recently, Behboodi et al. ([1]) considered this question in the context of finite direct products of commutative Noetherian local rings. In this paper, we continue their study by dropping the Noetherian condition.

1. Introduction

The study of rings over which modules in various classes are direct sums of cyclic modules has a long history. The first important contribution in this direction is due to Köthe [6] who considered rings over which all modules are direct sums of cyclic modules. Köthe showed that over an Artinian principal ideal ring, each module is a direct sum of cyclic modules. Furthermore, if a commutative Artinian ring has the property that all its modules are direct sums of cyclic modules, then it is necessarily a principal ideal ring. Later, Cohen and Kaplansky [3] obtained the following.

^{*}The research of the first author was in part supported by a grant from IPM (No. 90160034).

[†]*Key Words:* Cyclic modules; local rings; principal ideal rings.

[‡]2010 *Mathematics Subject Classification.* Primary 13C05, 13E05, 13F10, Secondary 13E10, 13H99.

[§]Corresponding author.

Result 1.1. (Cohen and Kaplansky, [3]) *If R is a commutative ring such that each R -module is a direct sum of cyclic modules, then R must be an Artinian principal ideal ring.*

An interesting natural question arises. Instead of considering rings for which *all* modules are direct sums of cyclic modules, we weaken this condition and study rings R for which it is assumed only that the *ideals* of R are direct sums of cyclic modules. The study of such commutative rings was initiated by Behboodi, Ghorbani and Moradzadeh-Dehkordi in [1]. In particular, they established the following theorem.

Result 1.2. ([1, Theorem 2.11]) *Let (R, \mathcal{M}) be a commutative Noetherian local ring, where \mathcal{M} denotes the unique maximal ideal of R . Then the following statements are equivalent:*

- (1) *Every ideal of R is a direct sum of cyclic R -modules.*
- (2) *$\mathcal{M} = Rw_1 \oplus \dots \oplus Rw_n$ where $n \geq 1$ and at most 2 of Rw_1, \dots, Rw_n are not simple.*
- (3) *There exists $n \geq 1$ such that every ideal of R is a direct sum of at most n cyclic R -modules.*
- (4) *Every ideal of R is a summand of a direct sum of cyclic R -modules.*

In this paper, we consider commutative local rings for which every ideal is a direct sum of cyclic modules (that is, we drop the Noetherian condition from [1]). In particular, we describe the ideal structure of such rings.

In the sequel, all rings are commutative with identity and all modules are unital. For a ring R , we denote (as usual) the set of prime ideals of R by $\text{Spec}(R)$. Also, $\text{Nil}(R)$ is the ideal of all nilpotent elements of R . We denote the classical Krull dimension of R by $\dim(R)$. Let X be either an element or a subset of R . The annihilator of X is the ideal $\text{Ann}(X) = \{a \in R \mid aX = 0\}$. A ring R is *local* in case R has a unique maximal ideal. In this paper (R, \mathcal{M}) will be a local ring with maximal ideal \mathcal{M} . An R -module N is called *simple* if $N \neq (0)$ and it has no submodules except (0) and N . An R -module M is a *semisimple* module if it is a direct sum of simple modules. Also, an R -module M is called a *homogenous semisimple* R -module if it is a direct sum of isomorphic simple R -modules, i.e., $\text{Ann}(M)$ is a maximal ideal of R .

It will be shown that if a local ring (R, \mathcal{M}) has the property that every ideal of R is a direct sum of cyclic R -modules, then $\dim(R) \leq 1$, $|\text{Spec}(R)| \leq 3$. Moreover, $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ where $x, y, w_\lambda \in R$ ($\lambda \in \Lambda$) and each Rw_λ is a simple R -module, $R/\text{Ann}(x)$, $R/\text{Ann}(y)$ are principal ideal rings and $\text{Spec}(R) \subseteq \{(0), \mathcal{M}, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda), Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\}$ (see Theorems 3.1 and 3.3). Also, we prove the following main theorem.

Result 1.3. (See Theorem 3.7) *For a local ring (R, \mathcal{M}) the following statements are equivalent:*

- (1) Every ideal of R is a direct sum of cyclic R -modules.
- (2) Every ideal of R is a direct sum of cyclic R -modules, at most two of which are not simple.
- (3) $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ where Λ is an index set, Rw_λ s ($\lambda \in \Lambda$) are simple R -modules and $x, y \in R$ are such that $R/\text{Ann}(x)$ and $R/\text{Ann}(y)$ are principal ideal rings.
- (4) Every ideal of R is a summand of a direct sum of cyclic R -modules.

Finally, some relevant examples and counterexamples are indicated in Section 4.

2. Preliminaries

We begin this section with the following result which states that to check whether every ideal in a ring is principal, it suffices to test only the prime ideals.

Lemma 2.1. (Attributed to I. M. Isaacs in [5, p. 8, Exercise 10]) *A commutative ring R is a principal ideal ring if and only if every prime ideal of R is a principal ideal.*

Lemma 2.2. (Kaplansky [4, Theorem 12.3]) *A commutative Noetherian ring R is a principal ideal ring if and only if every maximal ideal of R is principal.*

Also, by using Nakayama's lemma, we obtain the following two lemmas.

Lemma 2.3. *Let R be a ring and M be an R -module such that M is a direct sum of a family of finitely generated R -modules. Then Nakayama's lemma holds for M (i.e., for each $I \subseteq J(R)$, if $IM = M$, then $M = (0)$).*

Lemma 2.4. (See [1, Lemma 2.1]) *Let (R, \mathcal{M}) be a local ring and Rx be a nonzero summand of \mathcal{M} . Then Rx is a simple R -module if and only if $x^2 = 0$.*

Proof. Assume that $x^2 = 0$ and $\mathcal{M} = Rx \oplus L$ for some ideal L . Since $\mathcal{M}x = (Rx \oplus L)x = 0$, so Rx is simple. The proof of the converse is clear. \square

Proposition 2.5. *Let R be a ring. If every prime ideal of R is a direct sum of cyclic R -modules, then for each prime ideal P of R , the ring R/P is a principal ideal domain (PID). Consequently, $\dim(R) \leq 1$.*

Proof. Assume that every prime ideal of R is a direct sum of cyclic R -modules and $P \subset Q$ are prime ideals of R . Suppose that $Q = \bigoplus_{i \in I} Rx_i$ where I is an index set and $x_i \in R$ for each $i \in I$. There exists a $j \in I$ such that $x_j \notin P$. Since for each $i \in I$, $Rx_i Rx_j = (0) \subseteq P$ and P is prime, for each $i \neq j$, $Rx_i \subseteq P$ and so we conclude that Q/P is a principal ideal of R/P . Thus every prime ideal of the ring R/P is principal and hence by Lemma 2.1, R/P is a PID. Since this holds for all prime ideals P of R , we conclude that $\dim(R) \leq 1$. \square

The following two results from [1] are crucial to our investigation.

Lemma 2.6. ([1, Proposition 2.2]) *Let (R, \mathcal{M}) be a local ring and $\mathcal{M} = Rx \oplus Ry \oplus Rz \oplus K$ where K is an ideal of R , $0 \neq x, y, z \in R$ and Rx, Ry, Rz are not simple R -modules. Then the ideal $J := R(x + y) + R(x + z)$ is not a direct sum of cyclic R -modules.*

Lemma 2.7. ([1, Corollary 2.3]) *Let (R, \mathcal{M}) be a local ring. If every ideal of R is a direct sum of cyclic R -modules, then $\mathcal{M} = \bigoplus_{\lambda \in \Lambda} Rw_\lambda$ where Λ is an index set, $w_\lambda \in R$ for each $\lambda \in \Lambda$, and at most two of the Rw_λ s ($\lambda \in \Lambda$) are not simple.*

Lemma 2.8. (See [8, Proposition 3]) *Let R be a local ring and M an R -module. If $M = \bigoplus_{\lambda \in \Lambda} R/I_\lambda$ where each I_λ is an ideal of R , then every summand of M is also a direct sum of cyclic R -modules, each isomorphic to one of the R/I_λ .*

Lemma 2.9. *Let R be a ring and M a homogenous semisimple R -module. Then $M = \bigoplus_{\lambda \in \Lambda} Rw_\lambda$ where Λ is an index set, Rw_λ s are isomorphic simple R -modules and every submodule of M is also of the form $N = \bigoplus_{\gamma \in \Gamma} Rw'_\gamma$ where Γ is an index set with $|\Gamma| \leq |\Lambda|$ and Rw'_γ s are isomorphic simple R -modules.*

Proof. The proof is clear from the fact that $\text{Ann}(M) = P$ is a maximal ideal of R and M is an R/P -vector space. \square

We conclude this section with the following proposition from [1] that is an analogue of Invariant Basis Number (IBN) of free modules over commutative rings.

Lemma 2.10. ([1, Proposition 2.15]) *Let R be a ring. Then the following statements are equivalent:*

- (1) *R is a local ring.*
- (2) *If $\bigoplus_{i=1}^n Rx_i \cong \bigoplus_{j=1}^m Ry_j$ where $n, m \in \mathbb{N}$ and Rx_i, Ry_j are nonzero cyclic R -modules, then $n = m$.*
- (3) *If $\bigoplus_{i \in I} Rx_i \cong \bigoplus_{j \in J} Ry_j$ where I, J are index sets and Rx_i, Ry_j are nonzero cyclic R -modules, then $|I| = |J|$.*

3. Main Results

Theorem 3.1. *Let (R, \mathcal{M}) be a local ring such that every ideal of R is a direct sum of cyclic R -modules. Then $\dim(R) \leq 1$, $|\text{Spec}(R)| \leq 3$ and $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ where Λ is an index set, $x, y, w_\lambda \in R$ and each Rw_λ ($\lambda \in \Lambda$) is a simple R -module. Moreover,*

- (a) *If $x, y \in \text{Nil}(R)$, then $\text{Spec}(R) = \{\mathcal{M}\}$.*
- (b) *If $\mathcal{M} = Rz$ and $z \notin \text{Nil}(R)$, then $\text{Spec}(R) = \{(0), \mathcal{M}\}$.*

- (c) If \mathcal{M} is not cyclic, $x \notin \text{Nil}(R)$ and $y \in \text{Nil}(R)$, then $\text{Spec}(R) = \{\mathcal{M}, Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\}$.
(d) If \mathcal{M} is not cyclic, $x \in \text{Nil}(R)$ and $y \notin \text{Nil}(R)$, then $\text{Spec}(R) = \{\mathcal{M}, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\}$.
(e) If \mathcal{M} is not cyclic and $x, y \notin \text{Nil}(R)$, then

$$\text{Spec}(R) = \{\mathcal{M}, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda), Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\}.$$

Proof. By Proposition 2.5, $\dim(R) \leq 1$. Also, by Lemma 2.7, $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ where Λ is an index set, $x, y, w_\lambda \in R$ and Rw_λ s ($\lambda \in \Lambda$) are simple R -modules (i.e., $w_\lambda^2 = 0$ for each $\lambda \in \Lambda$ by Lemma 2.4). We consider the following four cases.

Case a: $x, y \in \text{Nil}(R)$. Since $w_\lambda^2 = 0$ for each $\lambda \in \Lambda$, we conclude that $\text{Nil}(R) = \mathcal{M}$ and so $\text{Spec}(R) = \{\mathcal{M}\}$.

Case b: $\mathcal{M} = Rz$ and $z \notin \text{Nil}(R)$. Then $\dim(R) = 1$. Let $P \in \text{Spec}(R) \setminus \{\mathcal{M}\}$. Since $P \subsetneq \mathcal{M} = Rz$, $P = Pz$ and so by Lemma 2.3, $P = (0)$. Thus by Lemma 2.1, R is a principal ideal domain and $\text{Spec}(R) = \{(0), \mathcal{M}\}$.

Case c: \mathcal{M} is not cyclic, $x \notin \text{Nil}(R)$ and $y \in \text{Nil}(R)$. Thus $\text{Nil}(R) \neq \mathcal{M}$ and so $\dim(R) = 1$. Let P be a prime ideal of R such that $P \subsetneq \mathcal{M}$. Since $w_\lambda^2 = 0$ for each $\lambda \in \Lambda$, we conclude that $Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda) \subseteq P$. Thus $P = (Rx \cap P) \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$. Since $\mathcal{M} \not\subseteq P$, it follows that $x \notin P$ and so $Rx \cap P = Px$. Thus $P = Px \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ and hence $Px = Px^2 = RxPx$, so by Lemma 2.3, $Px = 0$. Thus $P = Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$. Therefore, $\text{Spec}(R) = \{\mathcal{M}, Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\}$. Also, if \mathcal{M} is not cyclic, $x \in \text{Nil}(R)$ and $y \notin \text{Nil}(R)$, then similarly, we conclude that $\text{Spec}(R) = \{\mathcal{M}, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\}$.

The proof in case d is the same.

Case d: \mathcal{M} is not cyclic and $x, y \notin \text{Nil}(R)$. Thus $\text{Nil}(R) \neq \mathcal{M}$ and so $\dim(R) = 1$. Let P be a prime ideal of R such that $P \subsetneq \mathcal{M}$. Since $xy = 0$, $x \in P$ or $y \in P$. If $x \in P$, then $y \notin P$ and $P = Rx \oplus (P \cap Ry) \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$. As above, we have $Ry \cap P = Py = (0)$ and $P = Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$. Similarly, if $y \in P$, then $P = Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$. On the other hand, since $x, y \notin \text{Nil}(R)$, there exist $P_1, P_2 \in \text{Spec}(R) \setminus \{\mathcal{M}\}$ such that $x \in P_1, y \notin P_1$ and $x \notin P_2, y \in P_2$. Therefore, $\text{Spec}(R) = \{\mathcal{M}, Rx \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda), Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)\}$. \square

We can now state the following corollary. This result is an analogue of Kaplansky's theorem (see Lemma 2.2).

Cohen in [2], proved that if R is a commutative ring, Then R is Noetherin if and only if every prime ideal of R is finitely generated.

Corollary 3.2. *Let (R, \mathcal{M}) be a local ring such that every ideal of R is a direct sum of cyclic R -modules. Then \mathcal{M} is cyclic (resp. finitely generated) if and only if R is a principal ideal ring (resp. Noetherian ring).*

Proof. From Theorem 3.1, we see that if M is principal (resp. finitely generated) then the same holds for every prime ideal of R . The conclusion follows from Lemma 2.1 (resp. Cohen's theorem). \square

Next, we sharpen Corollary 2.3 of [1] (see Lemma 2.7).

Theorem 3.3. *Let (R, \mathcal{M}) be a local ring such that every ideal of R is a direct sum of cyclic R -modules. Then, with the notation of Theorem 3.1, both $R/\text{Ann}(x)$ and $R/\text{Ann}(y)$ are principal ideal rings.*

Proof. Using Theorem 3.1, we see that each prime ideal of $S := R/(Ry \oplus (\bigoplus_{\lambda \in \Lambda} R w_\lambda))$ is principal. Now Lemma 2.1 implies that S is a principal ideal ring. Since $Ry \oplus (\bigoplus_{\lambda \in \Lambda} R w_\lambda) \subseteq \text{Ann}(x)$, $R/\text{Ann}(x)$ is a homomorphic image of S and hence is a principal ideal ring. Similarly, $R/\text{Ann}(y)$ is a principal ideal ring. \square

Let us now outline the proof of the main theorem of this paper. We have divided the proof into a sequence of propositions.

Proposition 3.4. *Let (R, \mathcal{M}) be a local ring such that $\mathcal{M} = Rx \oplus L$ for some ideal L of R and some $x \in R$. Suppose further that $R/\text{Ann}(x)$ is a principal ideal ring. Then every nonzero element of R is of the form ax^n for some unit a and positive integer n .*

Proof. Assume that R is local with maximal ideal \mathcal{M} . Suppose further that $\mathcal{M} = Rx \oplus L$ for some ideal L of R and some $x \in R$ and that $R/\text{Ann}(x)$ is a principal ideal ring. By way of contradiction, assume that there is some nonzero element $r_1 \in Rx$ which is not of the form ax^n for any unit $a \in R$ and any positive integer n . Let $\pi : \mathcal{M} \rightarrow \mathcal{M}$ be the projection function onto the first coordinate (that is, $\pi(rx + l) := rx$ for $r \in R$ and $l \in L$). By our assumption on r_1 (and the fact that R is local), we have $r_1 = sx$ for some $s \in M$. But by the directness of the sum decomposition of M , it is clear that $r_1 = sx = \pi(s)x$. By our contradiction assumption, we conclude that $\pi(s) := r_2$ is not a unit. Hence $r_2 = tx$ for some $t \in M$ (since r_1 is, by assumption, not equal to a unit times a power of x). But then $r_2 = \pi(t)x$. Now set $r_3 := \pi(t)$. Continuing inductively, we obtain a sequence r_1, r_2, r_3, \dots of elements of R such that for every positive integer n , $r_n = r_{n+1}x$ (and recall that $r_1 \neq 0$). Now set $J := \{s_1 \in R : \text{there exist elements } s_2, s_3, s_4, \dots \in R \text{ such that } s_n = s_{n+1}x \text{ for every } n \in \mathbb{N}\}$. One checks at once that J is a nonzero ideal of R (since $r_1 \in J$). We claim that $J = Jx$. To see this, let $j_1 \in J$ be arbitrary. Then by definition of J , we see that $j_1 = j_2x$ for some $j_2 \in R$. It is clear from the definition of J that $j_2 \in J$. This shows that $J = Jx \subseteq Rx$. Since $R/\text{Ann}(x)$ is a principal ideal ring, we conclude that $J = Jx$ is finitely generated (cyclic, in fact). We now invoke Nakayama's lemma to conclude that $J = \{0\}$, a contradiction. \square

Proposition 3.5. *Let (R, \mathcal{M}) be a local ring such that $\mathcal{M} = Rx \oplus (\bigoplus_{\lambda \in \Lambda} R w_\lambda)$ where Λ*

is an index set, $Rw_\lambda s$ ($\lambda \in \Lambda$) are simple R -modules and $x \in R$ is such that $R/\text{Ann}(x)$ is principal ideal ring. Then every proper ideal of R is of the form $I = Rx' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_\gamma)$ where Γ is an index set with $|\Gamma| \leq |\Lambda|$, $Rw'_\gamma s$ ($\gamma \in \Gamma$) are simple R -modules and $x' \in R$ is such that $R/\text{Ann}(x')$ is a principal ideal ring.

Proof. Assume that I is a proper ideal of R and $L = \bigoplus_{\lambda \in \Lambda} Rw_\lambda$. Clearly, L is a homogenous semisimple R -module with $L^2 = (0)$. If $I \subseteq Rx$, then every ideal contained in I is principal, and we are done since $R/\text{Ann}(x)$ is a principal ideal ring). Also, if $I \subseteq L$, then by Lemma 2.9, I is a direct sum of at most $|\Lambda|$ simple modules. Thus we can assume that $I \not\subseteq Rx$, $I \not\subseteq L$ and $(0) \subsetneq I \subsetneq \mathcal{M}$. By Proposition 3.4, there exist $n \in \mathbb{N}$ and $l \in L$ such that $x^n + l \in I$. Among all such expressions, choose one, $x^{n_0} + l_0$, with n_0 minimal. We set $x' = x^{n_0}$ if $x^{n_0} \in I$, otherwise we set $x' = x^{n_0} + l_0$. Set

$$J := \{l \in L \mid ax + l \in I, \text{ for some } a \in R\}.$$

Then J is an ideal of R with $J \subseteq L$. It is easy to check that $Rx' \cap (I \cap J) = (0)$ and $Rx' \oplus (I \cap J) \subseteq I$. We will show that $I = Rx' \oplus (I \cap J)$. Assume that $u = ax^s + l \in I$ where $a \in (R \setminus \mathcal{M}) \cup \{0\}$, $s \in \mathbb{N}$ and $l \in L$. If $a = 0$, then $l \in I \cap J$ and so $u = l \in Rx' \oplus (I \cap J)$. Thus we can assume that $a \in R \setminus \mathcal{M}$. Therefore,

$$u - ax^{s-n_0}x' = (ax^s + l) - ax^{s-n_0}(x^{n_0} + l_0) = l + ax^{s-n_0}l_0 \in I.$$

Since l and l_0 contained in J , $l + ax^{s-n_0}l_0 \in J$. Hence it follows that $u \in Rx' \oplus (I \cap J)$. Therefore, $I = Rx' \oplus (I \cap J)$. By Lemma 2.9, $I \cap L$ is a direct sum of at most $|\Lambda|$ simple R -modules. Also, since $\mathcal{M}l_0 = (0)$, we conclude that $\text{Ann}(x^{n_0} + l_0) = \text{Ann}(x^{n_0})$. This implies that $Rx' \cong Rx^{n_0} \subseteq Rx$. Since $R/\text{Ann}(x)$ is a principal ideal ring, so $R/\text{Ann}(x')$ is also a principal ideal ring. \square

Proposition 3.6. Let (R, \mathcal{M}) be a local ring such that $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ where Λ is an index set, $Rw_\lambda s$ ($\lambda \in \Lambda$) are simple R -module and $x, y \in R$ are such that $R/\text{Ann}(x)$ and $R/\text{Ann}(y)$ are principal ideal rings. Then every ideal of R is one of the following forms:

- (i) $I = Rx' \oplus Ry' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_\gamma)$ where Γ is an index set with $|\Gamma| \leq |\Lambda|$, $Rw'_\gamma s$ are simple R -modules and $x', y' \in R$ are such that $R/\text{Ann}(x')$ and $R/\text{Ann}(y')$ are principal ideal rings.
- (ii) $I = Rz' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_\gamma)$ where Γ is an index set with $|\Gamma| \leq |\Lambda|$, $Rw'_\gamma s$ are simple R -modules and $z' \in R$ is such that every ideal of $R/\text{Ann}(z')$ is a direct sum of at most two cyclic ideals.

Proof. By Proposition 3.5, we can assume that Rx and Ry are not simple R -modules. Let $L = \bigoplus_{\lambda \in \Lambda} Rw_\lambda$. Clearly, R/Rx (resp. R/Ry) is a local ring with the maximal ideal

$\bar{\mathcal{M}} = \mathcal{M}/Rx \cong Ry \oplus L$ (resp. $\bar{\mathcal{M}} = \mathcal{M}/Ry \cong Rx \oplus L$). Assume that I is an ideal of R . If $I \subseteq Rx \oplus L$ (resp. $I \subseteq Ry \oplus L$), then $I \cong (I \oplus Ry)/Ry$ (resp. $I \cong (I \oplus Rx)/Rx$) and so by Proposition 3.5, $I = Rx' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_\gamma)$ where Γ is an index set with $|\Gamma| \leq |\Lambda|$, Rw'_γ ($\gamma \in \Gamma$) are simple R -modules and $x' \in R$ is such that $R/\text{Ann}(x')$ is a principal ideal ring. Thus we can assume that $I \not\subseteq Rx \oplus L$, $I \not\subseteq Ry \oplus L$ and $(0) \subsetneq I \subsetneq \mathcal{M}$.

By Proposition 3.4, every element of I is of the form $ax^s + by^t + l$ where $a \in (R \setminus \mathcal{M}) \cup \{0\}$, $b \in (R \setminus \mathcal{M}) \cup \{0\}$, $s, t \in \mathbb{N}$ and $l \in L$. Since $I \not\subseteq Rx \oplus L$, $I \not\subseteq Ry \oplus L$, it follows that there exist $e_1, e_2 \in I$ where $e_1 = a_1x^n + b_1y^t + l_1$, $e_2 = a_2x^s + b_2y^m + l_2$ for some $a_1, b_2 \in R \setminus \mathcal{M}$, $a_2, b_1 \in R$, $l_1, l_2 \in L$ and $n, s, t, m \in \mathbb{N}$. Thus $xe_1 = a_1x^{n+1} \in I$ and $ye_2 = b_2y^{m+1} \in I$. Since $a_1, b_2 \in R \setminus \mathcal{M}$, so $x^{n+1} \in I$ and $y^{m+1} \in I$. Suppose that n_0 (resp. m_0) is the smallest natural number such that there exists an element $x^{n_0} + l_1 \in I$ (resp. $y^{m_0} + l_2 \in I$) for some $l_1 \in L$ (resp. $l_2 \in L$). We set $x' = x^{n_0}$ if $x^{n_0} \in I$, otherwise we set $x' = x^{n_0} + l_1$. Also, we set $y' = y^{m_0}$ if $y^{m_0} \in I$, otherwise we set $y' = y^{m_0} + l_2$. Set

$$J := \{l \in L \mid ax + by + l \in I, \text{ for some } a, b \in R\}.$$

Then J is an ideal of R with $J \subseteq L$. One can easily see that $Rx' + Ry' + (I \cap J)$ is a direct sum and $Rx' \oplus Ry' \oplus (I \cap J) \subseteq I$. Now we proceed by cases.

Case 1: $I = Rx' \oplus Ry' \oplus (I \cap J)$. Then by Lemma 2.9, $I \cap J$ is a direct sum of at most $|\Lambda|$ simple R -modules. It follows that I is a direct sum of at most $|\Lambda| + 2$ cyclic modules. Also, if $x' \neq 0$, then $x^{n_0} \neq 0$ and so $\text{Ann}(x') = \text{Ann}(x^{n_0})$ since $\mathcal{M}l_1 = (0)$. It follows that $R/\text{Ann}(x')$ is a principal ideal ring (since $Rx^{n_0} \subseteq Rx$). We conclude similarly that either $Ry' = (0)$ or $R/\text{Ann}(y')$ is a principal ideal ring.

Case 2: $Rx' \oplus Ry' \oplus (I \cap J) \subsetneq I$. We claim that every element of $I \setminus (Rx' \oplus Ry' \oplus (I \cap J))$ is of the form $z = cx^{n_0-1} + dy^{m_0-1} + l$ where $c, d \in R \setminus \mathcal{M}$. Let $z = cx^s + dy^t + l \in I \setminus (Rx' \oplus Ry' \oplus (I \cap J))$ where $c, d \in (R \setminus \mathcal{M}) \cup \{0\}$ and $l \in L$. If $c = 0$, then $z = dy^t + l \in I$ and so $t \geq m_0$. If $t > m_0$, then $z = dy^{t-m_0}(y^{m_0} + l_2) + l \in Ry' \oplus (I \cap J)$, a contradiction. Thus $t = m_0$ and this implies that $z = dy' + (l - dl_2) \in Ry' \oplus (I \cap J)$, a contradiction. Thus $c \in R \setminus \mathcal{M}$. We conclude similarly that $d \in R \setminus \mathcal{M}$. Our next claim is that $s = n_0 - 1$ and $t = m_0 - 1$, for if not, to obtain a contradiction, we need to consider the following three subcases:

Subcase 1: $s < n_0 - 1$ or $t < m_0 - 1$. There is no loss of generality in assuming that $s < n_0 - 1$. Then $x^{n_0-1} = x^{n_0-1-s}c^{-1}z \in I$ which contradicts the minimality of n_0 .

Subcase 2: $s \geq n_0$ and $t \geq m_0$. Then $z = cx^{s-n_0}x' + dy^{t-m_0}y' - (cx^{s-n_0}l_1 - dy^{t-m_0}l_2) + l \in I$. Since $z - cx^{s-n_0}x' + dy^{t-m_0}y' \in I$, so $(cx^{s-n_0}l_1 - dy^{t-m_0}l_2) + l \in (I \cap J)$ and hence $z \in Rx' \oplus Ry' \oplus (I \cap J)$, a contradiction.

Subcase 3: $s \geq n_0$, $t = m_0 - 1$ or $s = n_0 - 1$, $t \geq m_0$. Without loss of generality

we can assume $s \geq n_0$ and $t = m_0 - 1$. Then $z = cx^{s-n_0}x' + dy^{m_0-1} - cx^{s-n_0}l_1 + l$ and so $z - cx^{s-n_0}x' = dy^{m_0-1} - cx^{s-n_0}l_1 + l \in I$. Thus $y^{m_0-1} + d^{-1}(l - cx^{s-n_0}l_1) \in I$ which contradicts the minimality of m_0 .

Therefore, we have $z = cx^{n_0-1} + dy^{m_0-1} + l$ where $c, d \in R \setminus \mathcal{M}$. Now let us fix arbitrarily $z' = cx^{n_0-1} + dy^{m_0-1} + l \in I \setminus (Rx' \oplus Ry' \oplus (I \cap J))$ where $c, d \in R \setminus \mathcal{M}$. Since $L^2 = (0)$, it is easy to check that $Rz' \cap (I \cap J) = (0)$. Now we claim that $I = Rz' \oplus (I \cap J)$. We note that $x' = xc^{-1}z' + l_1$ and since $l_1 = x' - xc^{-1}z' \in I$, so $l_1 \in I \cap J$. Hence $x' \in Rz' \oplus (I \cap J)$. We conclude similarly that $y' \in Rz' \oplus (I \cap J)$. Thus $Rx' \oplus Ry' \oplus (I \cap J) \subseteq Rz' \oplus (I \cap J) \subseteq I$. Suppose contrary to our claim, that $I \not\subseteq Rz' \oplus (I \cap J)$. Then there exists an element $u \in I \setminus (Rz' \oplus (I \cap J))$ and so $u \in I \setminus (Rx' \oplus Ry' \oplus (I \cap J))$. Therefore, $u = c'x^{n_0-1} + d'y^{m_0-1} + l'$ for some $c', d' \in R \setminus \mathcal{M}$ and $l' \in L$. Then

$$u - c'c^{-1}z' = (c'x^{n_0-1} + d'y^{m_0-1} + l') - c'c^{-1}(cx^{n_0-1} + dy^{m_0-1} + l) = d''y^{m_0-1} + l'' \in I$$

where $d'' = d' - c'c^{-1}d$ and $l'' = l' - c'c^{-1}l$. If $d'' = 0$, then $l'' \in (I \cap J)$ and so $u = c'c^{-1}z' + l'' \in Rz' \oplus (I \cap J)$, a contradiction. Also, if $d'' \in R \setminus \mathcal{M}$, then $y^{m_0-1} + d''^{-1}l'' \in I$ which contradicts the minimality of m_0 . We thus get $d'' \in \mathcal{M}$ and hence there exists $r \in R$ such that $d''y^{m_0-1} = ry^{m_0}$. Therefore,

$$u - c'c^{-1}z' = ry^{m_0} + l'' = ry' - rl_2 + l'' \in I,$$

and since $l'' - rl_2 \in I \cap J$, it follows that $u \in Rz' \oplus (I \cap J)$, a contradiction.

Therefore, we have $I = Rz' \oplus (I \cap J)$. By Lemma 2.9, $I \cap J$ is a direct sum of at most $|\Lambda|$ simple R -modules. It follows that I is a direct sum of at most $|\Lambda| + 1$ cyclic R -modules. We need only consider the structure of each ideal of $R/\text{Ann}(z')$. It is easy to check that $\text{Ann}(z') = \text{Ann}(x^{n_0-1}) \cap \text{Ann}(y^{m_0-1})$. Also, a trivial verification shows that

$$\text{Ann}(x^{n_0-1}) = H_1 \oplus Ry \oplus L \quad \text{and} \quad \text{Ann}(y^{m_0-1}) = Rx \oplus H_2 \oplus L$$

where

$$\begin{aligned} H_1 &= \{rx \in Rx \mid rx + sy + l \in \text{Ann}(x^{n_0-1}), \text{ for some } s \in R, l \in L\}, \\ H_2 &= \{sy \in Ry \mid rx + sy + l \in \text{Ann}(y^{m_0-1}), \text{ for some } r \in R, l \in L\}. \end{aligned}$$

Therefore, we conclude that $\text{Ann}(z') = H_1 \oplus H_2 \oplus L$. Put $\bar{R} = R/\text{Ann}(z')$. Then \bar{R} is a local ring with the maximal ideal $\bar{\mathcal{M}} := \mathcal{M}/\text{Ann}(z') \cong Rx/H_1 \oplus Ry/H_2$. It follows that $\bar{\mathcal{M}}$ is a direct sum of at most two cyclic \bar{R} -modules. Thus by Corollary 3.2, the ring \bar{R} is Noetherian. Therefore, by [1, Theorem 2.11], every ideal of \bar{R} is a direct sum of at most two cyclic modules (see Result 1.4), which completes the proof. \square

Now we are in a position to prove the main theorem. This theorem is an answer to the question “What is the class of local rings R for which every ideal is a direct sum of cyclic modules?”

Theorem 3.7. *Let (R, \mathcal{M}) be a local ring. Then the following statements are equivalent:*

- (1) *Every ideal of R is a direct sum of cyclic R -modules.*
- (2) *Every ideal of R is a direct sum of cyclic R -modules, at most two of which are not simple.*
- (3) *$\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} Rw_\lambda)$ where Λ is an index set, Rw_λ ($\lambda \in \Lambda$) are simple R -modules and $x, y \in R$ are such that $R/\text{Ann}(x)$ and $R/\text{Ann}(y)$ are principal ideal rings.*
- (4) *There exists an index set Λ such that every ideal of R is one of the following forms:*
 - (i) *$I = Rx' \oplus Ry' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_\gamma)$ where Γ is an index set with $|\Gamma| \leq |\Lambda|$ where Rw'_γ s are simple R -modules and $x', y' \in R$ such that $R/\text{Ann}(x')$, $R/\text{Ann}(y')$ are principal ideal rings.*
 - (ii) *$I = Rz' \oplus (\bigoplus_{\gamma \in \Gamma} Rw'_\gamma)$ where Γ is an index set with $|\Gamma| \leq |\Lambda|$, Rw'_γ s are simple R -modules and $z' \in R$ is such that every ideal of $R/\text{Ann}(z')$ is a direct sum of at most two principal ideals.*
- (5) *Every ideal of R is a summand of a direct sum of cyclic R -modules.*

Proof. (1) \Rightarrow (3). Follows from Theorem 3.3.

(3) \Rightarrow (4). Follows from Propositions 3.5 and 3.6.

(4) \Rightarrow (2), (2) \Rightarrow (1) and (1) \Rightarrow (5) are clear.

(5) \Rightarrow (1). Follows from Lemma 2.8. \square

In particular, the following corollary is an answer to the question “What is the class of local rings (R, \mathcal{M}) where \mathcal{M} is finitely generated and every ideal of R is a direct sum of cyclic modules?” Also, this result is an analogue of Kaplansky’s theorem (Lemma 2.2) (see also [1, Theorem 2.11]).

Corollary 3.8. *Let (R, \mathcal{M}) be a local ring. Then the following statements are equivalent:*

- (1) *\mathcal{M} is finitely generated and every ideal of R is a direct sum of cyclic R -modules.*
- (1) *R is Noetherian and every ideal of R is a direct sum of cyclic R -modules.*
- (2) *$\mathcal{M} = Rv_1 \oplus \dots \oplus Rv_n$ where $n \geq 1$, at most two of Rv_1, \dots, Rv_m are not simple and for each i the ring $R/\text{Ann}(v_i)$ is a principal ideal ring.*
- (3) *There exists $n \geq 1$ such that every ideal of R is of the form $I = Rv_1 \oplus \dots \oplus Rv_m$ where $m \leq n$ and at most two of Rv_1, \dots, Rv_m are not simple.*
- (4) *There exists $n \geq 1$ such that every ideal of R is a direct sum of at most n cyclic R -modules.*
- (5) *There exists $n \geq 1$ such that every ideal of R is a summand of a direct sum of at most n cyclic R -modules.*

Proof. The proof is straightforward by Corollary 3.2, Theorem 3.7 and [1, Theorem 2.11]. \square

Remark 3.9. Let $R = R_1 \times \cdots \times R_k$ where $k \in \mathbb{N}$ and each R_i is a nonzero ring. One can easily see that, the ring R has the property that whose ideals are direct sum of cyclic R -modules if and only if for each i the ring R_i has this property.

We are thus led to the following strengthening of Theorem 3.7.

Corollary 3.10. *Let $R = R_1 \times \cdots \times R_k$ where $k \in \mathbb{N}$ and each R_i is a local ring with maximal ideal \mathcal{M}_i ($1 \leq i \leq k$). Then the following statements are equivalent:*

- (1) *Every ideal of R is a direct sum of cyclic R -modules.*
- (2) *For each i , every ideal of R_i is a direct sum of cyclic R_i -modules, at most two of which are not simple.*
- (3) *For each i , $\mathcal{M}_i = R_i x \oplus R_i y \oplus (\bigoplus_{\lambda \in \Lambda_i} R_i w_\lambda)$ where Λ_i is an index set, $R_i w_\lambda$ ($\lambda \in \Lambda_i$) are simple R_i -modules and $x, y \in R_i$ are such that $R_i/\text{Ann}(x)$, $R_i/\text{Ann}(y)$ are principal ideal rings.*
- (4) *There exist index sets $\Lambda_1, \dots, \Lambda_k$ such that for each i , every ideal of R_i is of the forms $I = R_i x' \oplus R_i y' \oplus (\bigoplus_{\gamma \in \Gamma_i} R_i w'_\gamma)$ or $I = R_i z' \oplus (\bigoplus_{\gamma \in \Gamma_i} R_i w'_\gamma)$ where Γ_i is an index set with $|\Gamma_i| \leq |\Lambda_i|$, $R_i w'_\gamma$ ($\gamma \in \Gamma_i$) are simple R_i -modules $x', y', z' \in R_i$ such that $R_i/\text{Ann}(x')$, $R_i/\text{Ann}(y')$ are principal ideal rings and every ideal of $R_i/\text{Ann}(z')$ is a direct sum of at most two principal ideals.*
- (5) *Every ideal of R is a summand of a direct sum of cyclic R -modules.*

Proof. The proof is straightforward by Theorem 3.7 and Remark 3.9. \square

4. Examples

In this section some relevant examples and counterexamples are indicated. We begin with the following interesting example. In fact, the following example shows that the corresponding of the above Corollary 3.10, is not true in general for the case $R = \prod_{\lambda \in \Lambda} R_\lambda$ where Λ is an infinite index set and each R_λ is a local ring (even if for each $\lambda \in \Lambda$, $R_\lambda \cong \mathbb{Z}_2$).

Example 4.1. Let $R = \prod_{\lambda \in \Lambda} F_\lambda$ be a direct product of fields $\{F_\lambda\}_{\lambda \in \Lambda}$ where Λ is an infinite index set. Clearly, $I = \bigoplus_{\lambda \in \Lambda} F_\lambda$ is a non-maximal ideal of R . Thus there exists a maximal ideal P of R such that $I \subsetneq P$. It was shown by Cohen and Kaplansky [3, Lemma 1] that P is not a direct sum of principal ideals.

Let (R, \mathcal{M}) be a local ring. By Theorem 3.7, every ideal of R is a direct sum of cyclic R -modules if and only if $\mathcal{M} = Rx \oplus Ry \oplus (\bigoplus_{\lambda \in \Lambda} R w_\lambda)$ where Λ is an index set, $x, y, w_\lambda \in R$ ($\lambda \in \Lambda$) and (i) $R w_\lambda$ s are simple R -modules, and (ii) $R/\text{Ann}(x)$, $R/\text{Ann}(y)$ are

principal ideal rings. However, in this case, every ideal I of R has the structure of the form $I = Rx' \oplus Ry' \oplus (\bigoplus_{\gamma \in \Gamma} R w'_\gamma)$ where Γ is an index set with $|\Gamma| \leq |\Lambda|$, $R w'_\gamma$ ($\gamma \in \Gamma$) are simple R -modules and $x', y' \in R$, but the following example shows that the last property (ii) does not hold for all ideals of R , in general (even if R is Artinian and \mathcal{M} is two generated).

Example 4.2. Let F be a field, $n \geq 3$ and R be the F -algebra with generators x, y subject to the relations $x^n = y^n = xy = 0$ (i.e., $R \cong F[X, Y]/\langle X^n, Y^n, XY \rangle$). The ring R is a Noetherian local ring with maximal ideal $\mathcal{M} = Rx \oplus Ry$. Since $\mathcal{M}^n = (0)$, $\dim(R) = 0$ and so R is an Artinian local ring. Also, by Theorem 3.7, every ideal of R is a direct sum of at most two cyclic R -modules and also $R/\text{Ann}(x)$, $R/\text{Ann}(y)$ are principal ideal rings. Now let $I = Rz$ where $z = x + y$. We note that, if $I = \bigoplus_{i=1}^n Rz_i$ where $n \in \mathbb{N}$ and Rz_i are nonzero cyclic R -modules, then by Lemma 2.10, $n = 1$. Set $\bar{R} = R/\text{Ann}(z)$. Clearly $\text{Ann}(z) = Rx^{n-1} \oplus Ry^{n-1}$. Since $\bar{\mathcal{M}} := \mathcal{M}/\text{Ann}(z) \cong (Rx/Rx^{n-1}) \oplus (Ry/Ry^{n-1})$, it follows that the maximal ideal $\bar{\mathcal{M}}$ of \bar{R} is a direct sum of two nonzero cyclic \bar{R} -modules and hence by Lemma 2.10, $\bar{\mathcal{M}}$ is not principal, i.e., \bar{R} is not a principal ideal ring.

For each integer $n \geq 3$, there exists an Artinian local ring (R, \mathcal{M}) such that \mathcal{M} is a direct sum of n cyclic R -modules, but there exists a two generated ideal of R such that it is not a direct sum of cyclic R -modules (see [1, Example 3.1]). Also, the following example shows that there exists non-Noetherian rings R such that every prime ideal of R is a direct sum of cyclic R -modules, but some of the ideals of R are not direct sums of cyclic R -modules.

Example 4.3. Let F be a field and let R be the F -algebra with generators $\{x_i \mid i \in \mathbb{N}\}$ subject to the relations

$$x_1^3 = x_2^3 = x_3^3 = x_k^2 = 0, \quad k \geq 4 \text{ and } x_i x_j = 0 \text{ for all } i \neq j$$

(i.e., $R \cong F[\{X_i \mid i \in \mathbb{N}\}]/\langle X_1^3, X_2^3, X_3^3, X_k^2, X_i X_j \mid 4 \leq k \in \mathbb{N}, i \neq j \geq 1 \rangle$). Then R is a non-Noetherian local ring with maximal ideal $\mathcal{M} = \bigoplus_{i \in \mathbb{N}} Rx_i$. Thus the only prime ideal of R is a direct sum of cyclic R -modules since $\mathcal{M}^3 = (0)$. But by Lemma 2.6, the ideal $J = R(x_1 + x_2) + R(x_1 + x_3)$ is not a direct sum of cyclic R -modules.

By Theorem 3.1, if a local ring (R, \mathcal{M}) has the property that every ideal of R is a direct sum of cyclic R -modules, then $\dim(R) \leq 1$ and $|\text{Spec}(R)| \leq 3$. The following example more or less summarizes the overall situation for $\dim(R)$ and $|\text{Spec}(R)|$.

Example 4.4. Let F be a field and $n \in \mathbb{N}$. Consider the following rings:

- (1) $R_1 = F[[X]]$ (formal power series ring).
- (2) $R_2 = F[\{X_i \mid 1 \leq i \leq n\}]/\langle \{X_i X_j \mid 1 \leq i, j \leq n\} \rangle$.
- (3) $R_3 = F[\{X_i \mid i \in \mathbb{N}\}]/\langle \{X_i X_j \mid i, j \in \mathbb{N}\} \rangle$.

- (4) $R_4 = F[[X, Y]] / \langle XY, Y^2 \rangle$.
- (5) $R_5 = F[[\{X_i \mid i \in \mathbb{N}\}]] / \langle \{X_i X_j \mid i \neq j\} \cup \{X_i^2 \mid i \geq 2\} \rangle$.
- (6) $R_6 = F[[X, Y]] / \langle XY \rangle$.
- (7) $R_7 = F[[\{X_i \mid i \in \mathbb{N}\}]] / \langle \{X_i X_j \mid i \neq j\} \cup \{X_i^2 \mid i \geq 3\} \rangle$.

It is easy to check that all above rings are local. Let \mathcal{M}_i be the maximal ideal of R_i ($1 \leq i \leq 7$). Then we easily obtain the following:

- (1) R_1 is a Noetherian domain (in fact R_1 is a PID) with $\dim(R_1) = 1$, $\mathcal{M}_1 = \langle X \rangle$ and $\text{Spec}(R_1) = \{(0), \mathcal{M}_1\}$.
- (2) R_2 is a non-domain Artinian ring with $\dim(R_2) = 0$, $\mathcal{M}_2 = R_2 x_1 \oplus \dots \oplus R_2 x_n$ (where $x_i = X_i + \langle \{X_i X_j \mid 1 \leq i, j \leq n\} \rangle$) and $\text{Spec}(R_2) = \{\mathcal{M}_2\}$.
- (3) R_3 is a non-domain, non-Noetherian ring with $\dim(R_3) = 0$, $\mathcal{M}_3 = \bigoplus_{i=1}^{\infty} R_3 x_i$ (where $x_i = X_i + \langle \{X_i X_j \mid i, j \in \mathbb{N}\} \rangle$) and $\text{Spec}(R_3) = \{\mathcal{M}_3\}$.
- (4) R_4 is a non-domain, Noetherian ring with $\dim(R_4) = 1$, $\mathcal{M}_4 = R_4 x \oplus R_4 y$ (where $x = X + \langle XY, Y^2 \rangle$ and $y = X + \langle XY, Y^2 \rangle$) and $\text{Spec}(R_4) = \{\mathcal{M}_4, R_4 y\}$.
- (5) R_5 is a non-domain, non-Noetherian ring with $\dim(R_5) = 1$, $\mathcal{M}_5 = \bigoplus_{i=1}^{\infty} R_5 x_i$ (where $x_i = X_i + \langle \{X_i X_j \mid i \neq j\} \cup \{X_i^2 \mid i \geq 2\} \rangle$) and $\text{Spec}(R_5) = \{\mathcal{M}_5, \bigoplus_{i=2}^{\infty} R_5 x_i\}$.
- (6) R_6 is a non-domain, Noetherian ring with $\dim(R_6) = 1$, $\mathcal{M}_6 = R_6 x \oplus R_6 y$ (where $x = X + \langle XY \rangle$ and $y = Y + \langle XY \rangle$) and $\text{Spec}(R_6) = \{\mathcal{M}_6, R_6 x, R_6 y\}$.
- (7) R_7 is a non-domain, non-Noetherian ring with $\dim(R_7) = 1$, $\mathcal{M}_7 = \bigoplus_{i=1}^{\infty} R_7 x_i$ (where $x_i = X_i + \langle \{X_i X_j \mid i \neq j\} \cup \{X_i^2 \mid i \geq 3\} \rangle$) and

$$\text{Spec}(R_7) = \{\mathcal{M}_7, \bigoplus_{i=2}^{\infty} R_7 x_i, R_7 x_1 \oplus (\bigoplus_{i=3}^{\infty} R_7 x_i)\}.$$

Now by Theorem 3.7, one can easily see that for each i ($1 \leq i \leq 7$) the ring R_i has the property that every ideal is a direct sum of cyclic ideals.

References

- [1] M. Behboodi, A. Ghorbani, A. Moradzadeh-Dehkordi, *Commutative Noetherian local rings whose ideals are direct sums of cyclic modules*, J. Algebra, 345 (2011) 257-265.
- [2] I. S. Cohen, *Rings with restricted minimum condition*, Duke Math. J., 17 (1950), 27-42.
- [3] I. S. Cohen and I. Kaplansky, *Rings for which every module is a direct sum of cyclic modules*, Math. Z., 54 (1951) 97-101.
- [4] I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc., 66 (1949) 464-491.
- [5] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.
- [6] G. Köthe, *Verallgemeinerte abelsche gruppen mit hyperkomplexen operatorenring*, Math. Z., 39 (1935) 31-44.

- [7] R. Y. Sharp, Steps In Commutative Algebra, Second edition, London Mathematical Society Student Texts, 51. Cambridge University Press, Cambridge, 2000.
- [8] R. B. Jr., Warfield, *A Krull-Schmidt theorem for infinite sums of modules*, Proc. Amer. Math. Soc., 22 (1969) 460-465.